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On the Summation of Divergent Series. *By* PROFESSOR DE MORGAN.

IN the last Number I gave the most elementary view I could arrive at of Arbogast's method of development. In the communication following I saw that Mr. Peter Gray had referred to Stirling's theorem; and this suggested that it might be useful to give, by means of common algebra only, an account of the two most important cases of summation of many terms of a divergent series.

The method I use depends upon a theorem which, simple as it is, I cannot find mentioned by any writer. Stated in the language of infinites, it would be—The infinite sums of two diverging series are in the ratio of their last terms. Stated in the language of limits, it is—If $a_1+a_2+a_3+\ldots$ and $b_1+b_2+b_3+\ldots$ be diverging series, then the ratio of $a_1+\ldots+a_n$ to $b_1+\ldots+b_n$, as n increases without limit, continually approaches to the value of a_n to b_n. And this, be it understood, whether the diverging series have their terms increasing or diminishing.

Some persons, even though well-informed mathematicians, confess to having but a cloudy idea of terms which diminish, and diminish without limit, mounting up to any sum we please. The difficulty arises from not looking at both sides of the question: we may get *as much as we please;* but we are at liberty to take *as*

many terms as we want. It is the old story of the pendulum which revolted in disgust at the number of times it would have to tick before the end of the year, and was brought to reason by being reminded that it would have just as many seconds to tick in.

The series $1+\frac{1}{2}+\frac{1}{3}+\ldots$ is of this kind. Parcel it out as follows:—

$$1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+(\tfrac{1}{9}+\ldots+\tfrac{1}{16})+\ldots$$

Each parcel is obviously more than half a unit. If then we want to exceed a million of millions, we have nothing to do but to sum two millions of millions of these lots. I mention this common proof that I may give another which I never found in a book, though it is not mine. It is well known that when $a-b+c-d+\ldots$ consists of terms diminishing without limit, the series is convergent, with a limit between a and $a-b$. Now

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots \text{ is } \begin{array}{l} 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots \\ \quad +1 \quad\; +\tfrac{1}{2}+\ldots \end{array}$$

And if it be S, we have $S=a+S$, where a is finite. Hence S is infinite.* An objection may be made of a kind which would have counted for nothing in the last century—and which would be met by saying we have shown that the series cannot be convergent, for then its infinitely small deferred remainder would be a finite quantity.

Carry the terms only as far as $(2n)^{-1}$, and the preceding method, since $1-\frac{1}{2}+\frac{1}{3}-\ldots$ is $\log 2$,† gives the following:—The greater n is, the more nearly

$$\frac{1}{n+1}+\ldots\ldots+\frac{1}{2n}=\log 2.$$

This is just what would be true if the equation $1+\ldots+n^{-1}=\log n$ were nearly true when n is great: and we have here the easiest presumption of the connection between $1+\ldots+n^{-1}$ and $\log n$.

* This ingenious proof was given me, 37 years ago, by a pupil of the age of 13, whose mathematical power was singularly in advance of his years. Of many things as worthy of remark in one so young, I only remember what is here given. Time and thought have developed this boy into Professor Sylvester, whose inventive power, in everything to which his taste has led him, places him in the highest rank.

The divergence of the series was first noticed and proved by John Bernoulli, and another proof was given by James Bernoulli. Both are much more difficult than that by collection into lots each greater than half a unit. I do not know who first gave this.

† In this *Journal*, supported by contributors who have constantly to think of the common logarithm, it is very common to distinguish the Naperean logarithm when it is used. But it is now so well established, in algebraical writing, that $\log x$ shall mean the Naperean logarithm, that it would be a good thing if, without further mention, the common logarithm were always denoted by $c. \log x$.

I now proceed to the theorem stated. Since the series are divergent, if we begin at a_m and b_m instead of a_1 and b_1, m being any given number, however great, $a_m+a_{m+1}+\ldots$ and $b_m+b_{m+1}+\ldots$, the same number of terms being taken out of both, have the same limiting ratio as

$$(a_1+\ldots+a_{m-1})+a_m+\ldots \text{ and } (b_1+\ldots+b_{m-1})+b_m+\ldots$$

This because two quantities which increase without limit have a limiting ratio which is not altered by adding any given quantities to both. Let l be the limit of $a_n : b_n$, and take m so great that at and after $n=m$, $a_n : b_n$ shall lie between $l\pm a$. Then, by a common theorem of arithmetic, $a_m : b_m$, $a_{m+1} : b_{m+1}$, &c., all lying between $l\pm a$, so does $a_m+a_{m+1}+\ldots$ divided by $b_m+b_{m+1}+\ldots$ This last fraction then, and consequently $(a_1+\ldots):(b_1+\ldots)$, has a limit between $l\pm a$, however small a may be; its limit is therefore l, the limit of $a_n : b_n$.

Next, let c_n be a quantity which increases without limit with n; or better, let $c_\infty=\infty$, a symbol which those who choose can translate into the language of limits. Then $c_1+(c_2-c_1)+(c_3-c_2)+\ldots$ is a divergent series; for the sum of n terms is c_n. That is, $a_1+\ldots+a_n$ divided by c_n has the same limit as a_n divided by c_n-c_{n-1}; or the equation

$$a_1+a_2+\ldots+a_n=\frac{c_n}{c_n-c_{n-1}}a_n$$

has sides which approach without limit to a ratio of equality as n increases without limit. If then we can contrive that a_n and c_n-c_{n-1} shall approach to a ratio of equality, we have a perpetual approach to truth in $a_1+\ldots+a_n=c_n$.

The process here divides as follows. Those who understand the integral calculus can be shown that c_n may be $\int a_n dn$: those who do not must be content to have the result of the integral calculus placed before them and verified.

Let $a_n=n^{-1}$; then c_n may be $\log n$: let $a_n=\log n$; then c_n may be $n\log n-n$. For in the first case

$$\frac{a_n}{c_n-c_{n-1}}=\frac{n^{-1}}{\log n-\log(n-1)}=\frac{n^{-1}}{-\log(1-n^{-1})}=\frac{n^{-1}}{n^{-1}+\frac{1}{2}n^{-2}+\ldots},$$

of which the limit is unity. In the second case

$$\frac{a_n}{c_n-c_{n-1}}=\frac{\log n}{n\log n-n-(n-1)\log(n-1)+n-1}.$$

Of this the denominator is

$$\log(n-1)-1+\{-n\log(1-n^{-1}) \text{ or } 1+\tfrac{1}{2}\frac{1}{n}+\ldots\},$$

and the limit of the fraction is unity.

Let us then assume, on trial,

$$1+\tfrac{1}{2}+\ldots+\frac{1}{n}=\log n+\mathrm{A}+\frac{\mathrm{B}}{n}+\frac{\mathrm{C}}{n^2}+\frac{\mathrm{D}}{n^3}+\ldots$$

Change n into $n+1$, and subtract, and we have

$$\frac{1}{n+1}=\log\left(1+\frac{1}{n}\right)+\mathrm{B}\left(\frac{1}{n+1}-\frac{1}{n}\right)+\mathrm{C}\left(\frac{1}{(n+1)^2}-\frac{1}{n^2}\right)+\mathrm{D}\left(\frac{1}{(n+1)^3}-\frac{1}{n^3}\right)+\ldots$$

$$\frac{1}{n}-\frac{1}{n^2}+\frac{1}{n^3}-\frac{1}{n^4}+\ldots=\frac{1}{n}-\tfrac{1}{2}\frac{1}{n^2}+\tfrac{1}{3}\frac{1}{n^3}-\tfrac{1}{4}\frac{1}{n^4}+\ldots$$

$$+\mathrm{B}\left(-\frac{1}{n^2}+\frac{1}{n^3}-\frac{1}{n^4}+\ldots\right)+\mathrm{C}\left(-\frac{2}{n^3}+\frac{3}{n^4}-\ldots\right)+\mathrm{D}\left(-\frac{3}{n^4}+\ldots\right)+\ldots$$

which is satisfied by

$$-1=-\tfrac{1}{2}-\mathrm{B},\ 1=\tfrac{1}{3}+\mathrm{B}-2\mathrm{C},\ -1=-\tfrac{1}{4}-\mathrm{B}+3\mathrm{C}-3\mathrm{D},\ \&c.$$
$$\mathrm{B}=\tfrac{1}{2},\ \mathrm{C}=-\tfrac{1}{12},\ \mathrm{D}=0,\ \&c.$$

This method may be easily carried further, and the result is

$$1+\tfrac{1}{2}+\ldots+\frac{1}{n}=\log n+\mathrm{A}+\frac{1}{2n}-\tfrac{1}{6}\frac{1}{2n^2}+\tfrac{1}{30}\frac{1}{4n^4}$$
$$-\tfrac{1}{42}\frac{1}{6n^6}+\tfrac{1}{30}\frac{1}{8n^8}-\ldots$$

By the method employed, it is clear that if this equation be true for any one value of n, it is true for the preceding and following values; for we have so constructed the second side of the equation (call it ϕn) as to satisfy

$$\phi(n+1)=\phi n+\frac{1}{n+1}.$$

Take A so as to satisfy this equation when $n=10$; this gives

$$2{\cdot}9289683=\mathrm{A}+\log 10+\frac{1}{20}-\frac{1}{1200}+\frac{1}{1200000}-\ldots$$

which gives, log 10 being 2·3025851, A=·5772157. More accurately

$$\mathrm{A}={\cdot}57721{,}56649{,}01532{,}86060{,}65\ldots$$

This constant, which I usually denote by γ, has that sort of importance which attaches to what are known as π and ε; that is, 3·14159... and 2·71828...

Let us now take

$$\log 1+\log 2+\ldots+\log n=n\log n-n+\mathrm{A}+\frac{\mathrm{B}}{n}+\ldots$$

As before, change n into $n+1$, and subtract: we have then

$$\log(n+1)=(n+1)\log(n+1)-n-1-n\log n+n$$
$$+B\left(\frac{1}{n+1}-\frac{1}{n}\right)+C\left(\frac{1}{(n+1)^2}-\frac{1}{n^2}\right)+\ldots$$
$$0=n\log\left(1+\frac{1}{n}\right)-1+B\left(\frac{1}{n+1}-\frac{1}{n}\right)+\ldots$$

Develope as before, and we find that we cannot produce an identical equation: out of the first term comes $-\frac{1}{2}n^{-1}$, the only term having n^{-1}. But when we look at $n\log n-n$, we see that if, instead of $n\log n-n$, we write $n\log n-n+N\log n$, where N is any constant, we still have approach to equality between $\log 1+\ldots+\log n$ and $n\log n-n+N\log n$; and this is all that is demanded by our original theorem. Let us then assume

$$\log 1+\ldots+\log n=n\log n-n+N\log n+A+\frac{B}{n}+\ldots$$

Proceeding as before, we have

$$0=n\log\left(1+\frac{1}{n}\right)-1+N\log\left(1+\frac{1}{n}\right)+B\left(\frac{1}{n+1}-\frac{1}{n}\right)+\ldots$$

After development we see that this equation is identically satisfied if

$$-\tfrac{1}{2}+N=0,\ \tfrac{1}{3}-\tfrac{1}{2}N-B=0,\ -\tfrac{1}{4}+\tfrac{1}{3}N+B-2C=0,$$
$$\tfrac{1}{5}-\tfrac{1}{4}N-B+3C-3D=0;$$
$$\text{or } N=\tfrac{1}{2},\ B=\tfrac{1}{12},\ C=0,\ D=-\tfrac{1}{360},\ \&c.$$

By this, carried further, we have

$$\log 1+\ldots+\log n=n\log n-n+\tfrac{1}{2}\log n+A+\tfrac{1}{6}\cdot\frac{1}{2n}-\tfrac{1}{30}\frac{1}{3.4n^3}$$
$$+\tfrac{1}{42}\frac{1}{5.6n^5}-\ldots$$

If we were to take the trouble of determining A by assuming, as before, $n=10$, and calculation, we should find $A=\log\sqrt{(2\pi)}$. But this may be taken for granted until the reader becomes acquainted with Wallis's theorem, as follows:—The greater n is made, the more nearly is

$$\frac{\pi}{2}=\frac{4}{3}\cdot\frac{16}{15}\cdot\frac{36}{35}\cdot\ldots\ldots\cdot\frac{4n^2}{4n^2-1},$$
$$\text{or } \frac{2.4.6\ldots.2n}{1.3.5\ldots.2n-1}=\sqrt{(n+\tfrac{1}{2})\pi}.$$

Let n be taken exceedingly great, so that all which follows A may be rejected as exceedingly small: that is,

$$\log 1 + \ldots . + \log n = n \log n - n + \tfrac{1}{2} \log n + \mathrm{A}$$

is as nearly true as we please. This gives

$$\begin{aligned}
1.2.3 \ldots .\; n &= n^{n+\frac{1}{2}} \varepsilon^{\mathrm{A}-n} \\
1.2.3 \ldots . 2n &= (2n)^{2n+\frac{1}{2}} \varepsilon^{\mathrm{A}-2n} \\
2.4.6 \ldots . 2n &= 2^n n^{n+\frac{1}{2}} \varepsilon^{\mathrm{A}-n} \\
1.3.5 \ldots . 2n-1 &= (2n)^n \sqrt{2}\, \varepsilon^{-n} \\
\frac{2.4.6 \ldots . 2n}{1.3.5 \ldots . 2n-1} &= \sqrt{\frac{n}{2}} \cdot \varepsilon^{\mathrm{A}} = \sqrt{(n+\tfrac{1}{2})\pi}
\end{aligned}$$

The greater n, the more nearly is this equation true: it becomes true, then, when n is made infinite; that is, $\varepsilon^{\mathrm{A}} = \surd(2\pi)$. And thus we have

$$1.2.3 \ldots . n = \surd(2\pi) . n^{n+\frac{1}{2}} \varepsilon^{-n} . \varepsilon^{\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \ldots .}$$

This approaches to truth, if we stop at what is written, as n increases; but it is very near the truth at the beginning. If $n=1$, we have

$$1 = \surd(2\pi) . \varepsilon^{-1} . \varepsilon^{\cdot 0813492}, \quad \surd(2\pi) = \varepsilon^{\cdot 9186508},$$

$$\text{or } \pi = \tfrac{1}{2} \varepsilon^{1 \cdot 8373016}.$$

This gives $\pi = 3{\cdot}1398$ instead of $3{\cdot}1416$.

There is a point about the demonstrations above which will not be quite clear at first to the young algebraist. We have assumed forms on one side of the equation: how do we know those forms are correct? As follows:—Let there be a series $\phi 1 + \phi 2 + \ldots + \phi n$. Suppose we find a function, ψn, which satisfies the equation $\psi(n+1) = \phi(n+1) + \psi n$. Let $\phi 1 + \ldots + \phi n$ be $\psi n + \chi n$. Then

$$\begin{aligned}
\phi 1 + \phi 2 + \ldots + \phi n + \phi(n+1) &= \psi(n+1) + \chi(n+1) \\
= \psi n + \chi n \quad + \phi(n+1) &= \psi(n+1) + \chi n.
\end{aligned}$$

Hence $\chi(n+1) = \chi n$; or χn is not changed by changing n into $n+1$. Step by step we find that if n be integer, χn is not changed by changing n into any other integer. Accordingly, since n is integer in every case we use, χn is, throughout our problem, a constant: that is, $\phi 1 + \ldots + \phi n$ differs from ψn by a constant. Now, in our demonstrations we assume a form for ψn which, by determination of constants, we find we can make to satisfy $\psi(n+1) - \psi n = \phi(n+1)$; and we have the constant A, which, not determined in the verification of this last equation, is subsequently determined.

The theorem of which the preceding contains particular cases is accessible to those who have an elementary knowledge of the integral calculus. Take the series $\phi 0 + \ldots + \phi n$: assume the form

$$\int \phi n dn + \mathrm{B}\phi n + \mathrm{C}\phi' n + \mathrm{D}\phi'' n + \mathrm{E}\phi''' n + \ldots .$$

for ψn. Can we determine B, C, &c., so that $\psi(n+1)-\psi n = \phi(n+1)$? If so, by help of the arbitrary constant in $\int\phi n dn$ we can make good the equality of $\phi 1+\ldots+\phi n$ and ψn. Change n into $n+1$. Then, by Taylor's theorem, $\int\phi n dn$ becomes $\int\phi n dn + \phi n + \phi' n.\frac{1}{2} + \ldots$, ϕn becomes $\phi n + \phi' n + \phi'' n.\frac{1}{2} + \ldots$, $\phi' n$ becomes $\phi' n + \phi'' n + \phi''' n.\frac{1}{2} + \ldots$ It will be convenient to write C : 2, D : 2.3, E : 2.3.4, &c., for C, D, E, &c.; and also to write subscript numerals for fractional divisors: thus, M_4 may mean M : 2.3.4, 1_3 may mean 1 : 2.3. This being agreed on, in $\psi n = \int\phi n dn + B_1\phi n + C_2\phi' n + D_3\phi'' n + E_4\phi''' n + \ldots$ write $n+1$ for n, and subtract ψn from $\psi(n+1)$. We have

$$\psi(n+1)-\psi n = \phi n + (1_2+B_1)\phi' n + (1_3+1_2B_1+C_2)\phi'' n$$
$$+(1_4+1_3B_1+1_2C_2+D_3)\phi''' n + (1_5+1_4B_1+1_3C_2+1_2D_3+E_4)\phi^{\text{iv}} n + \ldots$$

This is $\phi(n+1)$ if the coefficients of ϕn, &c., be 1, 1_1, 1_2, 1_3, &c.; so that we have the following results:—

$$1_2+B_1=1_1,\quad B=\tfrac{1}{2},\quad 1_3+1_2B_1+C_2=1_2,\quad \frac{1}{2.3}+\tfrac{1}{2}.\tfrac{1}{2}+\frac{C}{2}=\tfrac{1}{2},$$
$$\text{or } C=\tfrac{1}{6}.$$
$$1_4+1_3B_1+1_2C_2+D_3=1_3 \text{ gives } D=0,$$
$$1_5+1_4B_1+1_3C_2+1_2D_3+E_4=1_4 \text{ gives } E=-\tfrac{1}{30}.$$

Similarly, it will be found that $F=0$, $G=\frac{1}{42}$, $H=0$, $I=-\frac{1}{30}$, $K=0$, $L=\frac{5}{66}$, &c. And we have

$$\phi 0+\ldots+\phi n = A + \int\phi n dn + \tfrac{1}{2}\phi n + \tfrac{1}{6}\frac{\phi' n}{2} - \tfrac{1}{30}\frac{\phi''' n}{2.3.4} + \tfrac{1}{42}\frac{\phi^{\text{v}} n}{2.3\ldots6}$$
$$-\tfrac{1}{30}\frac{\phi^{\text{vii}} n}{2.3\ldots8} + \tfrac{5}{66}\frac{\phi^{\text{ix}} n}{2.3\ldots10} - \ldots$$

The fractions $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, &c., are the celebrated* *numbers of Bernoulli.* When A is not otherwise known, it may be determined by an instance, as hereinbefore shown. When the series is convergent, A is the sum *ad infinitum,* by which it is sometimes known. Though $\phi 0$ is made the first term, any other commencement might have been chosen; the constant A, properly found, sets the whole right.

As an example, we find that

$$1+\frac{1}{1+r}+\ldots+\frac{1}{1+nr} = A + \frac{1}{r}\log(1+nr) + \tfrac{1}{2}\frac{1}{1+nr}$$
$$-\tfrac{1}{6}\frac{r}{2(1+nr)^2} + \tfrac{1}{30}\frac{r^3}{4(1+nr)^4} - \ldots$$

* Todhunter, *Hist. of Probability,* p. 65. This work will be very useful to students who want to make a sound and complete preparation for the profession of an actuary.

Make $n=0$, and $A=\frac{1}{2}+\frac{1}{6}\frac{r}{2}-\frac{1}{30}\frac{r^3}{4}+\ldots$

Hence the value of n years annuity at simple interest, r per £1, is

$$\frac{1}{1+r}+\ldots+\frac{1}{1+nr}=\frac{1}{r}\log(1+nr)-\frac{1}{2}\left(1-\frac{1}{1+nr}\right)+\frac{1}{6}\frac{r}{2}\left(1-\frac{1}{(1+nr)^2}\right)$$
$$-\frac{1}{30}\frac{r^3}{4}\left(1-\frac{1}{(1+nr)^4}\right)+\frac{1}{42}\frac{r^5}{6}\left(1-\frac{1}{(1+nr)^6}\right)-\frac{1}{30}\frac{r^7}{8}\left(1-\frac{1}{(1+nr)^8}\right)$$
$$+\frac{5}{66}\frac{r^9}{10}\left(1-\frac{1}{(1+nr)^{10}}\right).$$

Multiply by r, for r write n^{-1}, and we have

$$\frac{1}{n+1}+\ldots+\frac{1}{2n}=\log 2-\frac{1}{4n}+\frac{1}{6}\frac{1}{2n^2}\left(1-\frac{1}{2^2}\right)-\frac{1}{30}\frac{1}{4n^4}\left(1-\frac{1}{2^4}\right)+\ldots$$

Add this to the formula already found for $1+\ldots+\frac{1}{n}$, and we see, as it ought to be, that the result is that formula with n changed into $2n$.

On a Table for the Formation of Logarithms and Anti-Logarithms to Twelve Places. By Peter Gray, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

Part IV.—*History of the Method.*

So far as I am aware, the earliest method for the formation of logarithms, in which it was proposed to resolve the number whose logarithm is required into factors *by a continuous process,* is one which was published by Mr. Manning in the *Philosophical Transactions* for 1806. My acquaintance with Mr. Manning's method is derived, not from the original paper, which I have not seen, but from a reprint of it, "nearly as it stands," in Young's *Elementary Essay on the Computation of Logarithms.** Mr. Manning applies his method to the formation of Naperian logarithms only; but it is equally applicable to the formation of common logarithms. I repeat, in accordance with this method, an example I have already more than once given.

Required the logarithm of π.

The following is the process:—

* Second edition, London, 1835, pp. 67–79.